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M. Sc. (Sem. - IV) Examination September - 2025

Mathematics (Paper - PGMTH:401) Functional Analysis - II

Time: 3 Hours] **[Total Marks: 70**

સૂચના : / Instructions

(1) ઉપરોક્ત દર્શાવેલ વિગતો ઉત્તરવહી પર અવશ્ય લખવી.
Fill up strictly the above details on your answer book.
(2) Answer all questions
(3) Figures to the right indicates marks
(4) Follow usual notations and conventions

Seat No.:

Student's Signature

Q. 1. Attempt Any Two: **(14)**

a. Prove that every bounded linear function f on a Hilbert space H can be uniquely represented in term of inner product, i.e. $f(x) = \langle x, z \rangle$, where z depends on f is uniquely determined by f and has $\| f \| = \| z \|$.

b. Let H_1 and H_2 be two Hilbert spaces and $T : H_1 \to H_2$ be a bounded linear operator. Then prove that the Hilbert adjoint operator T^* *of* T, an operator from H_2 to H_1 such that $\langle Tx, y \rangle = \langle x, T^* y \rangle$, $\forall x \in H_1, y \in H_2$, exists, is unique and is a bounded linear operator with norm $\| T^* \| = \| T \|$.

c. Let H be a Hilbert space and M is orthogonal set in H then prove that M is total if and only if for all non-zero $x \in H$ Parseval's relation $\Sigma_k |\langle x, e_k \rangle|^2 = \|x\|^2$ holds.

Q. 2. Attempt Any Two: **(14)**

a. Prove that the adjoint operator $T^\times$ of a bounded linear operator $T : X \to Y$, defined as $(T^\times g)(x) = g(Tx) = f(x)$, $\forall x \in X$, $g \in Y'$ and $f \in X'$ is linear, bounded and $\| T^\times \| = \| T \|$.

b. Let f be linear functional defined on a subspace Z of a normed space X, then there exists a bounded linear functional $\hat{f}$ on X, an extension of f from Z to X with norm $\| \hat{f} \|_X = \| f \|_Z$, where $\| \hat{f} \|_X = Sup_{\substack{x \in x \\ \|x\| = 1}} |\hat{f}(x)|$ and $\| f \|_Z = Sup_{\substack{x \in z \\ \|x\| = 1}} |f(x)|$.

c. Write Zorn's lemma. Prove that Every vector space $X \neq \{0\}$ has Hammel basis.

Q. 3. Attempt Any Two: **(14)**

a. Let H_1 and H_2 be two Hilbert spaces and $S: H_1 \to H_2$ *and* $T : H_1 \to H_2$ be bounded linear operators and α be any scalar, then prove that

i. $\langle T^* y, x \rangle = \langle y, Tx \rangle$

ii. $(S+T)^* = S^* + T^*$

iii. $(\alpha T)^* = \overline{\alpha} T^*$

iv. $(T^*)^* = T$

v. $\| TT^* \| = \| T^* T \| = \| T \|^2$

vi. $T^* T = 0 \Rightarrow T = 0$

vii. $(ST)^* = T^* S^*$, where $H_1 = H_2$

b. Let X and Y be two normed spaces and $B(X, Y)$, the set of all bounded linear operators X into Y then show that $(S+T) \in B(\text{X}, Y)$, $(S+T)^{\times} = S^{\times} + T^{\times}$ and $(\propto T)^{\times} = \propto T^{\times}$.

c. Prove that every Hilbert space H is reflexive.

Q. 4. Attempt Any Two: (14)

a. State and prove Uniform boundedness theorem.

b. If a metric space $X \neq \emptyset$ is complete, then prove that it is non-meager in itself. Hence if $X \neq \emptyset$ is complete and $X = \cup_{k=1}^{\infty} A_k$ then prove that at least one A_k contains a nonempty open subset.

c. Let (x_n) be a weakly convergent sequence in a normed space X, i.e. $x_n \xrightarrow{w} x$ then prove that

i. The weak limit x *of* (x_n) is unique

ii. Every subsequence (x_{n_k}) *of* (x_n) converges weakly to x.

iii. The sequence $\|(x_n)\|$ is bounded.

Q. 5. Attempt Any Two: (14)

a. Let (X, d) be a non-empty complete metric space and let $T : X \to X$ be a contraction mapping on X. Then prove that T has precisely one fixed point.

b. Let X and Y be Banach spaces and $T : D(T) \to Y$ a closed linear operator, where $D(T) \subset X$. Then prove that if $D(T)$ is closed in X, then operator T is bounded.

c. State and prove Bounded inverse theorem.
